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The Capacity of the Product Channel

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Given any pair of arbitrary alphabet channels satisfying coding theorem and its strong converse. Then the coding theorem and its strong converse also holds for the product channel and the capacity of the product channel equals the sum of the capacities of the components. Wyner derives this result in (Wyner, 1966) by means of list decoding methods (which obviously have no strong connection with this kind of problem). Here a simpler proof of the statement above is presented which uses only 'maximal code estimate' (see Wolfowitz, 1964) and a converse estimate.

Let the triple (X, F, M) consist of a set $X \neq \Phi$, a σ-field F on X and a nonempty set of probabilities $M = \{p\}$ on the measure space (X, F). M may be considered as the set of all transition probabilities for a channel and X as the set of all output symbols (for a fixed time).

A finite ϵ-code of M $(0 < \epsilon < 1)$ is a finite sequence $\{(p^i, E^i)\}_{1 \leqq i \leqq N}$ where the $p^i \in M$, $E^i \in F$, the E^i are pairwise disjoint and $p^i(E^i) > \epsilon$ $(1 \leqq i \leqq N)$. (The E^i are the decoding sets.) Call N the *length* of the finite code and set $N(M, \epsilon) := \sup \{N : N \text{ is length of an } \epsilon\text{-code of } M\}$.

With this definition $1 \leqq N(M, \epsilon) \leqq \infty$ holds $(0 < \epsilon < 1)$.

Let (X_v, F_v, M_v) $(v = 1, 2)$ be given as above (not necessarily copies of each other) and let $(X_1 \times X_2, F_1 \times F_2)$ denote the product of the measure spaces (X_1, F_1) and (X_2, F_2). Furthermore, let $M_1 \times M_2 := \{p_1 \times p_2 : p_1 \in M_1, p_2 \in M_2\}$ on $(X_1 \times X_2, F_1 \times F_2)$.

We will now derive an inequality relating $N(M_1, \cdot)$, $N(M_2, \cdot)$ and $N(M_1 \times M_2, \cdot)$ that leads directly to the statement in the summary.

First we have

$$N(M_1 \times M_2, \epsilon_1\epsilon_2) \geqq N(M_1, \epsilon_1)N(M_2, \epsilon_2)(0 < \epsilon_1, \epsilon_2 < 1). \quad (1)$$

(For let $\{(p_1^i, E_1^i)\}_{1 \leqq i \leqq N_1}$ be an ϵ_1-code for M_1 and $\{(p_2^j, E_2^j)\}_{1 \leqq j \leqq N_2}$

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be an ϵ_2-code for M_2 then $\{(p_1^i \times p_2^j, E_1^i \times E_2^j)\}_{1 \leq i \leq N_1, 1 \leq j \leq N_2}$ is an $\epsilon_1\epsilon_2$-code for $M_1 \times M_2$).

Next we upper estimate $N(M_1 \times M_2, \epsilon)$. Let $\{(p^i, E^i)\}_{1 \leq i \leq N}$ be a finite ϵ-code of $M_1 \times M_2$. The p^i are product probabilities $p^i = p_1^i \times p_2^i$.

Let

$$q: = q_1 \times q_2: = \left(\frac{1}{N} \sum_{i=1}^{N} p_1^i\right) \times \left(\frac{1}{N} \sum_{i=1}^{N} p_2^i\right). \tag{2}$$

Then for arbitrary $S_1, S > 0$ holds

$$\begin{aligned} S_1 S_2\, q(E^i) &\geqq p^i\left(\left\{\frac{dp^i}{dq} \leqq S_1 S_2\right\} \cap E^i\right) \\ &\geqq q^i(E^i) - p^i\left\{\frac{dp^i}{dq} > S_1 S_2\right\} > \epsilon - p^i\left\{\frac{dp^i}{dq} > S_1 S_2\right\} \\ &> \epsilon - p_1^i\left\{\frac{dp_1^i}{dq_1} > S_1\right\} - p_2^i\left\{\frac{dp_2^i}{dq_2} > S_2\right\}. \end{aligned}$$

Hence

$$\begin{aligned} S_1 S_2 \frac{1}{N} &\geqq S_1 S_2 \frac{1}{N} \sum_{i=1}^{N} q(E^i) \\ &> \epsilon - \frac{1}{N} \sum_{i=1}^{N} p^i\left\{\frac{dp^i}{dq_1} > S_1\right\} - \frac{1}{N} \sum_{i=1}^{N} p_2^i\left\{\frac{dp_2^i}{dq_2} > S_2\right\}. \end{aligned} \tag{3}$$

Remark. The probability q in (2) can be used efficiently to prove converses of the coding theorem for the nonstationary cases without memory (see Augustin, 1966).

The maximal code estimate (see Wolfowitz, 1964) gives

$$N(M_v, \bar{\epsilon}_v) > S_v\left[\frac{1}{N} \sum_{i=1}^{N} p_v^i\left\{\frac{dp_v^i}{dq_v} > S_v\right\} - \bar{\epsilon}_v\right] \tag{4}$$
$$(0 < \bar{\epsilon}_v < 1) \qquad (v = 1, 2).$$

Combining (3) and (4) results in

$$S_1 S_2 \frac{1}{N} > \epsilon - (\bar{\epsilon}_1 + \bar{\epsilon}_2) - \frac{N(M_1, \bar{\epsilon}_1)}{S_1} - \frac{N(M_2, \bar{\epsilon}_2)}{S_2}. \tag{5}$$

Assume now $\bar{\epsilon}_1 + \bar{\epsilon}_2 = (1 - c)\epsilon$ where $0 < c < 1$ and furthermore that $N(M_1, \bar{\epsilon}_1), N(M_2, \bar{\epsilon}_2) < \infty$ (otherwise (7) is trivially true) and put

$$S_v = \frac{3N(M_v, \bar{\epsilon}_v)}{c\epsilon} \qquad (v = 1, 2)$$

Then (5) yields

$$\frac{9}{c^2\epsilon^2} N(M_1, \bar{\epsilon}_1) N(M_2, \bar{\epsilon}_2) > N \frac{c\epsilon}{3} \quad \text{and} \tag{6}$$

we obtain together with (1):

$$\begin{aligned} N(M_1, \epsilon_1) N(M_2, \epsilon_2) &\leqq N(M_1 \times M_2, \epsilon) \\ &\leqq \left(\frac{3}{c\epsilon}\right)^3 N(M_1, \bar{\epsilon}_1) N(M_2, \bar{\epsilon}_2) \end{aligned} \tag{7}$$

where $\epsilon \leqq \epsilon_1\epsilon_2$ and $\bar{\epsilon}_1 + \bar{\epsilon}_2 < (1 - c)\epsilon$ $(0 < \epsilon_1, \epsilon_2, \epsilon, \bar{\epsilon}_1, \bar{\epsilon}_2 < 1)$.

Let a pair of channels be given and may $M_v(t)$ be the representation of the channel number v $(v = 1, 2)$ for the time interval $[0, t]$ (where $M_v(t)$ is defined similarly as earlier).

The product channel for $[0, t]$ may be represented by $M_1(t) \times M_2(t)$. Suppose that coding theorem and its strong converse holds for the channels $\{M_v(t)\}_{1 \leqq t < \infty}$ $(v = 1, 2)$ that is:

$$R_v : = \lim_{t \to \infty} (1/t) \ln N(M_v(t), \epsilon)$$

exists for all ϵ is finite and is independent of ϵ $(0 < \epsilon < 1)$ $(v = 1, 2)$. (R_v is the capacity of the channel $\{M_v(t)\}_{1 \leqq t < \infty}$.) Then

$$(1/t) \ln N(M_1(t_1) \times M_2(t), \epsilon) \xrightarrow{t \to \infty} R_1 + R_2 \qquad (0 < \epsilon < 1).$$

This follows immediately from (7).

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